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Maximum likelihood type estimation for discretely observed CIR model with small α -stable noises¹

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Abstract. A maximum likelihood type estimation of the drift and volatility coefficient parameters in the CIR type model driven by α -stable noises is studied when the dispersion parameter $\varepsilon \rightarrow 0$ and the discrete observations frequency $n \rightarrow \infty$ simultaneously.

1 Introduction

In mathematical finance, the classical Cox-Ingersoll-Ross (CIR) model describes the evolution of interest rates. It specifies that the instantaneous interest rate follows the stochastic differential equation (SDE):

$$dx_\varepsilon(t) = (a'_1 - a'_2 x_\varepsilon(t))dt + a'_3 \varepsilon \sqrt{x_\varepsilon(t)} dB(t), \quad (1.1)$$

where $\varepsilon, a'_1, a'_2, a'_3$ are strictly positive constants and $\{B_t : t \geq 0\}$ is a standard Brownian motion.

It is well-known that many financial processes exhibit discontinuous sample paths and heavy tailed properties (e.g. certain moments are infinite). These features cannot be captured by the CIR model. It is natural to replace the driving Brownian motion by an α -stable process; see [2] for the application of α -stable processes in finance. In this paper we are interested in the following stable driven CIR-type model:

$$dy_\varepsilon(t) = (a_1 - a_2 y_\varepsilon(t))dt + a_3 \varepsilon y_\varepsilon(s)^{1/q} dz_0(t), \quad y_\varepsilon(0) = x_0 \geq 0, \quad (1.2)$$

where $a_1, a_3 \geq 0$, $q > 0$, $a_2 \in \mathbb{R}$ are constants, and $\{z_0(t) : t \geq 0\}$ is a spectrally positive stable Lévy process with index $\alpha \in (1, 2)$ and Lévy measure $\mu(dz) := z^{-1-\alpha} 1_{\{z>0\}} dz$. By [12, Corollary 4.3], there is a pathwise unique positive strong solution $\{y_\varepsilon(t) : t \geq 0\}$ to (1.2) as $\frac{1}{q} + \frac{1}{\alpha} \geq 1$. In the case of $q = \alpha$, the solution is a particular form of the continuous-state branching processes with immigration (see [4, p.3]), which is also called the stable CIR model (see [11]). If $a_1 = a_2 = 0$, the solution can be treated as a critical branching process with population dependent branching rate by [23].

Assume that the unknown quantity in (1.2) are the parameters a_1, a_2, a_3 . The type of data considered in this paper is discrete observations at n regularly spaced time points $t_k = k/n$ on the fixed interval $[0, 1]$, that is $(y_\varepsilon(t_k))_{1 \leq k \leq n}$. The purpose of this paper is to study the maximum likelihood estimator for the true value of $\mathbf{a} := (a_1, a_2, a_3)$ based on these observations with small dispersion ε and large sample size n . To be precise, the type of asymptotics considered is when $\varepsilon = \varepsilon_n$ goes to 0 and n goes to ∞ simultaneously. The scheme of observations usually arises from

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some applied problems such as the identification of a real deterministic dynamic system with small random perturbations. We refer to [22] for an application of small dispersion asymptotics to contingent claim pricing.

The parameters estimation for discretely observed stochastic processes driven by small Brownian motion has been studied by several authors; see e.g. [19, 5, 20, 21]. The asymptotics distributions of the estimators based on a Gaussian approximation to the transition density (see [9]) are normal under certain conditions on $\varepsilon = \varepsilon_n$ and n ; see e.g. [19], where $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} (\varepsilon \sqrt{n})^{-1} < \infty$.

Recently, a number of papers have been devoted to small volatility asymptotics for the parameter estimation in the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation of discretely observed Lévy driven SDEs has been studied by many authors; see e.g. [13, 16, 15]. For the SDE (1.2), where the jumps are state-dependent and the jump term is *non-Lipschitz*, the asymptotics properties of the conditional least squares estimators and the weighted conditional least squares estimators of the drift parameters (a_1, a_2) were given in [11] based on low frequency observations, and the asymptotics behavior of the least squares estimator of the parameter a_1 (or a_2) was established in [17] under high frequency observations and small dispersion.

In this paper we employ a maximum likelihood type method to obtain an estimator for the parameter $\mathbf{a} = (a_1, a_2, a_3)$ in (1.2). To overcome the difficulty that the joint density of the sample $\{y_\varepsilon(t) : t \in [0, 1]\}$ is not tractable, we deal with by *using stable distributions to approximate the density*. It follows from (1.2) that

$$y_\varepsilon(t_k) = y_\varepsilon(t_{k-1}) + a_1 \Delta t_k - a_2 \int_{t_{k-1}}^{t_k} y_\varepsilon(s) ds + a_3 \varepsilon \int_{t_{k-1}}^{t_k} y_\varepsilon(s-)^{1/q} dz_0(s),$$

where $\Delta t_k = t_k - t_{k-1} = 1/n$. Then one can use the Euler scheme (see e.g. [7], which studied for a SDE driven by a Lévy process) to get the approximation

$$y_\varepsilon(t_k) \approx y_\varepsilon(t_{k-1}) + a_1 \Delta t_k - a_2 y_\varepsilon(t_{k-1}) \Delta t_k + a_3 \varepsilon \Delta t_k^{1/\alpha} y_\varepsilon(t_{k-1})^{1/q} z_k,$$

where z_1, z_2, \dots, z_n are independent stable random variables with the same distribution as $z_0(1)$. So as Δt_k , the distance between observations, is small, it may suggest that, conditioned on $y_\varepsilon(t_{k-1})$, the distribution of this random variable $y_\varepsilon(t_k) - y_\varepsilon(t_{k-1}) - a_1 \Delta t_k + a_2 y_\varepsilon(t_{k-1}) \Delta t_k$ may be close to that of $a_3 \varepsilon \Delta t_k^{1/\alpha} y_\varepsilon(t_{k-1})^{1/q} z_k$ in ceratin sense. Inspired by this, we can define a *likelihood type function* of $(y_\varepsilon(t_k))_{0 \leq k \leq n}$ by

$$L_{\varepsilon,n}(\mathbf{a}) := \prod_{k=1}^n [a_3 \varepsilon n^{-1/\alpha} y_\varepsilon(t_{k-1})^{1/q}]^{-1} p(Y_{\varepsilon,n,k}(\mathbf{a})),$$

where $p(x)$ is the density function of $z_0(1)$ and

$$\begin{aligned} Y_{\varepsilon,n,k}(\mathbf{a}) &:= [y_\varepsilon(t_k) - y_\varepsilon(t_{k-1}) - a_1 \Delta t_k + a_2 y_\varepsilon(t_{k-1}) \Delta t_k] \cdot [a_3 \varepsilon \Delta t_k^{1/\alpha} y_\varepsilon(t_{k-1})^{1/q}]^{-1} \\ &= [y_\varepsilon(t_k) - y_\varepsilon(t_{k-1}) - a_1/n + a_2 y_\varepsilon(t_{k-1})/n] \cdot [a_3 \varepsilon n^{-1/\alpha} y_\varepsilon(t_{k-1})^{1/q}]^{-1}. \end{aligned} \quad (1.3)$$

This likelihood type function may be a bit like the joint density of $(y_\varepsilon(t_k))_{0 \leq k \leq n}$ as Δt_k is small enough. Now we define the *log likelihood type function* of $(y_\varepsilon(t_k))_{0 \leq k \leq n}$ by

$$\tilde{U}_{\varepsilon,n}(\mathbf{a}) := \log L_{\varepsilon,n}(\mathbf{a}) = \sum_{k=1}^n \log p(Y_{\varepsilon,n,k}(\mathbf{a})) - n \log a_3 - q^{-1} \sum_{k=1}^n \log y_\varepsilon(t_{k-1}) - n \log(\varepsilon n^{-1/\alpha}).$$

Let $\hat{\mathbf{a}}_{\varepsilon,n} := (\hat{a}_{1,\varepsilon,n}, \hat{a}_{2,\varepsilon,n}, \hat{a}_{3,\varepsilon,n})$ be the *maximum likelihood type estimator* defined by $\tilde{U}_{\varepsilon,n}(\hat{\mathbf{a}}_{\varepsilon,n}) = \sup_{\mathbf{a} \in \bar{\mathbf{A}}} \tilde{U}_{\varepsilon,n}(\mathbf{a})$, where $\bar{\mathbf{A}}$ is the closure of an open set defined in Section 2. Such approximation is usually called an Euler-Maruyama approximation in the classical CIR model defined by (1.1); see e.g. [10, Section 9.1] and [18, Section 4.2.2].

It is obvious that $\hat{\mathbf{a}}_{\varepsilon,n}$ is also a maximum point of $U_{\varepsilon,n}$ defined by

$$U_{\varepsilon,n}(\mathbf{a}) := \sum_{k=1}^n \log p(Y_{\varepsilon,n,k}(\mathbf{a})) - n \log a_3, \quad U_{\varepsilon,n}(\hat{\mathbf{a}}_{\varepsilon,n}) = \sup_{\mathbf{a} \in \bar{\mathbf{A}}} U_{\varepsilon,n}(\mathbf{a}). \quad (1.4)$$

Our *main result* of this paper, Theorem 2.3, gives a consistent, asymptotically normal and asymptotically efficient estimator $\hat{\mathbf{a}}_{\varepsilon,n}$ of \mathbf{a} under the conditions $\varepsilon = \varepsilon_n \rightarrow 0$, $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} (\varepsilon n^{1/\alpha-1})^{-1} < \infty$, which is consistent with the corresponding assertion in [19, Theorem 1] if $\alpha = q = 2$. The proof is established in Section 3. An auxiliary lemma and the proof of Lemma 3.1 are presented in Section 4.

2 Main result

Before stating the main result of this paper, we give some notations. We always assume that all random elements are defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbf{P})$ satisfying the usual hypotheses. Let $C(\mathbb{R})$ be the space of continuous functions on \mathbb{R} . For $\mathbb{R}_+^2 := [0, \infty) \times [0, \infty)$ define $C(\mathbb{R}_+^2)$ similarly. For $f, g \in C(\mathbb{R})$ write $\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx$. For any integer $n \geq 1$ let $C^n(\mathbb{R})$ be the subset of $C(\mathbb{R})$ with continuous derivatives up to the n th order. Set $\|x\| = \sup_{t \in [0,1]} |x(t)|$. We use “ \xrightarrow{p} ” and “ \xrightarrow{d} ” to denote the convergence of random variables in probability and in distribution, respectively. Let \mathbf{a} be the parameter and $\bar{\mathbf{a}} := (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ the true value of \mathbf{a} . For $t \in [0, 1]$ define $y_0(t) = x_0 e^{-\bar{a}_2 t} + \bar{a}_1 \int_0^t e^{-\bar{a}_2(t-s)} ds$. Put

$$U(\mathbf{a}) = \int_0^1 dt \int_{\mathbb{R}} p(x) \log p(Y_0(\mathbf{a}, t, x)) dx - \log a_3, \quad (2.1)$$

where $Y_0(\mathbf{a}, t, x) := m_0 Y(\mathbf{a}, t) + \bar{a}_3 a_3^{-1} x$ and $Y(\mathbf{a}, t) := (\bar{a}_1 - a_1) a_3^{-1} y_0(t)^{-1/q} + (a_2 - \bar{a}_2) a_3^{-1} y_0(t)^{1-1/q}$. Define the matrices $\mathbf{V}(\mathbf{a}) := (\frac{\partial^2 U(\mathbf{a})}{\partial a_i \partial a_j})$ and

$$\Sigma := \begin{pmatrix} v_1 m^{0,2} & -v_1 m^{1,2} & v_2 m^{0,1} \\ -v_1 m^{1,2} & v_1 m^{2,2} & -v_2 m^{1,1} \\ v_2 m^{0,1} & -v_2 m^{1,1} & v_3 \end{pmatrix},$$

where $v_1 := \int_{\mathbb{R}} |p'(x)|^2 / p(x) dx$, $v_2 := \int_{\mathbb{R}} x |p'(x)|^2 / p(x) dx$, $v_3 := \int_{\mathbb{R}} x^2 |p'(x)|^2 / p(x) dx - 1$, and $m^{i,j} := \int_0^1 y_0(t)^{i-\frac{j}{q}} dt$ for $i, j \geq 0$. We give the conditions on the initial value $x_0 = y_\varepsilon(0)$ and the true value of the parameters.

Condition 2.1 *Neither of the following conditions hold: (i) $x_0 = \bar{a}_1 / \bar{a}_2$ and $\bar{a}_2 \neq 0$; (ii) $\bar{a}_1 = 0$ and $\bar{a}_2 = 0$; (iii) $x_0 = \bar{a}_1 = 0$.*

Observe that $y_0(t) > 0$ for all $t > 0$ under Condition 2.1. Since $\lim_{\varepsilon \rightarrow 0} \|y_\varepsilon - y_0\| = 0$ \mathbf{P} -a.s. by [17, Proposition 3.2], $y_\varepsilon(t) > 0$ for all $t \in [0, 1]$ as ε small enough. This makes sure that (1.3) is well defined. It is easy to see that $\frac{\partial U(\bar{\mathbf{a}})}{\partial a_i} = 0$ ($i = 1, 2, 3$) and $\mathbf{V}(\bar{\mathbf{a}}) = -\bar{a}_3^{-2} \Sigma$. Then $\bar{\mathbf{a}}$ is a local maximum point of $U(\mathbf{a})$ under Condition 2.1 by Lemma 4.1 in Appendix. In the following we state the conditions on the domain \mathbf{A} and the relationship between n and ε .

Condition 2.2 (i) Let \mathbf{A} be an open bounded convex subset of $[0, \infty) \times \mathbb{R} \times [0, \infty)$ and $\bar{\mathbf{A}}$ denote its closure set. Suppose that $\bar{\mathbf{A}} \cap (\mathbb{R}^2 \times \{0\}) = \emptyset$ and $\bar{\mathbf{a}} \in \mathbf{A}$ is the only maximum point of $U(\mathbf{a})$ on $\bar{\mathbf{A}}$. (ii) Suppose that $\varepsilon := \varepsilon_n$ and $\lim_{n \rightarrow \infty} m_{\varepsilon,n} = m_0 < \infty$, where $m_{\varepsilon,n} := \varepsilon^{-1} n^{\frac{1}{\alpha}-1}$.

Theorem 2.3 Assume that Conditions 2.1–2.2 hold. Then as $n \rightarrow \infty$,

$$\hat{\mathbf{a}}_{\varepsilon,n} \xrightarrow{p} \bar{\mathbf{a}}, \quad (2.2)$$

$$\mathbf{S}_{\varepsilon,n} := (v_{\varepsilon,n}(\hat{a}_{1,\varepsilon,n} - \bar{a}_1), v_{\varepsilon,n}(\hat{a}_{2,\varepsilon,n} - \bar{a}_2), \sqrt{n}(\hat{a}_{3,\varepsilon,n} - \bar{a}_3)) \xrightarrow{d} N(\mathbf{0}, \bar{a}_3^2 \boldsymbol{\Sigma}^{-1}), \quad (2.3)$$

where $v_{\varepsilon,n} := m_{\varepsilon,n} \sqrt{n} = \varepsilon^{-1} n^{\frac{1}{\alpha}-\frac{1}{2}}$ and $\mathbf{0} := (0, 0, 0)$.

3 Proof of Theorem 2.3

It follows from (1.2) that

$$y_\varepsilon(t_k) - y_\varepsilon(t_{k-1}) = \frac{\bar{a}_1}{n} - \bar{a}_2 \int_{t_{k-1}}^{t_k} y_\varepsilon(s) ds + \varepsilon \bar{a}_3 \int_{t_{k-1}}^{t_k} y_\varepsilon(s-)^{1/q} dz_0(s). \quad (3.1)$$

Together with (1.3) one derives that for $\mathbf{a} = (a_1, a_2, a_3)$, $n, k \geq 1$ and $\varepsilon > 0$,

$$\begin{aligned} Y_{\varepsilon,n,k}(\mathbf{a}) &= (\bar{a}_1 - a_1) a_3^{-1} m_{\varepsilon,n} n \int_{t_{k-1}}^{t_k} y_\varepsilon([ns]/n)^{-1/q} ds \\ &\quad + a_3^{-1} m_{\varepsilon,n} n \int_{t_{k-1}}^{t_k} [a_2 y_\varepsilon([ns]/n)^{1-1/q} - \bar{a}_2 y_\varepsilon(s) y_\varepsilon([ns]/n)^{-1/q}] ds \\ &\quad + \bar{a}_3 a_3^{-1} n^{1/\alpha} \int_{t_{k-1}}^{t_k} [y_\varepsilon(s-) y_\varepsilon([ns]/n)^{-1}]^{1/q} dz_0(s) \\ &=: (\bar{a}_1 - a_1) a_3^{-1} m_{\varepsilon,n} N_{\varepsilon,n,k} + a_3^{-1} m_{\varepsilon,n} \bar{M}_{\varepsilon,n,k}(a_2) + \bar{a}_3 a_3^{-1} K_{\varepsilon,n,k}, \end{aligned} \quad (3.2)$$

where $[x]$ denotes the largest integer not greater than x , and $M_{\varepsilon,n,k} = n \int_{t_{k-1}}^{t_k} y_\varepsilon([ns]/n)^{1-1/q} ds$ and $K_{n,k} = n^{1/\alpha} [z(t_k) - z(t_{k-1})]$. Before showing the proof of Theorem 2.3, we state the following lemma, which will be proved in Appendix.

Lemma 3.1 Suppose that Conditions 2.1–2.2 hold. Let $H \in C^1(\mathbb{R})$ and $B \in C(\mathbb{R}_+^2)$ satisfy

$$\sup_{x \geq 0} |H'(x)| + \sup_{x < 0} |(-x)^{-\gamma} H'(x)| < \infty \quad (3.3)$$

and $|B(x_1, y_1) - B(x_2, y_2)| \leq C_k [|x_1 - x_2| + |y_1 - y_2|]$ for all $x_1, x_2, y_1, y_2 \in [0, k]$ and $k \geq 1$, where $\gamma > 1$ and $C_k > 0$ are constants. Then as $n \rightarrow \infty$,

$$\sup_{\mathbf{a} \in \bar{\mathbf{A}}} \left| \frac{1}{n} \sum_{k=1}^n H(Y_{\varepsilon,n,k}(\mathbf{a})) B(M_{\varepsilon,n,k}, N_{\varepsilon,n,k}) - \int_0^1 B_t dt \int_{\mathbb{R}} p(x) H(Y_0(\mathbf{a}, t, x)) dx \right| \xrightarrow{p} 0. \quad (3.4)$$

Moreover, if $\bar{H} \in C^1(\mathbb{R})$ satisfying (3.3) with H replaced by \bar{H} and $\langle \bar{H}, p \rangle = \langle H, p \rangle = 0$, then

$$n^{-1/2} \sum_{k=1}^n \left[H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) B(M_{\varepsilon,n,k}, N_{\varepsilon,n,k}) + \bar{H}(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) \right] \xrightarrow{d} N(0, \eta_0^2) \quad (3.5)$$

as $n \rightarrow \infty$, where $\eta_0^2 := \int_0^1 \langle (B_t H + \bar{H})^2, p \rangle dt$ and $B_t := B(y_0(t)^{1-1/q}, y_0(t)^{-1/q})$.

Lemma 3.2 For each $k \geq 0$, there are constants $c_k, c'_k > 0$ so that $p^{(k)}(x) \sim c_k x^{-\alpha-1-k}$ and $p^{(k)}(-x) \sim c'_k \xi^{\frac{2-\alpha+2k}{2\alpha}} e^{-\xi}$ as $x \rightarrow \infty$, where $\xi = (\alpha-1)(x/\alpha)^{\alpha/(\alpha-1)}$.

Proof. The proofs of these two assertions follow immediately from the arguments in [24, Theorem 2.5.1] and [24, Theorem 2.5.2], respectively. \square

For $x \in \mathbb{R}$ define $H_0(x) = p'(x)/p(x)$, $H_1(x) = [p''(x)p(x) - |p'(x)|^2]/p(x)^2$, $H_2(x) = xH_1(x) + H_0(x)$ and $H_3(x) = x^2H_1(x) + 2xH_0(x) + 1$. For $i, j = 1, 2, 3$ let $U_{\varepsilon,n}^i(\mathbf{a}) = \frac{\partial U_{\varepsilon,n}(\mathbf{a})}{\partial a_i}$ and $U_{\varepsilon,n}^{i,j}(\mathbf{a}) = \frac{\partial^2 U_{\varepsilon,n}(\mathbf{a})}{\partial a_i \partial a_j}$. For $1 \leq i_1, j_1 \leq 2$ and $(i_2, j_2) \in \{(1, 3), (2, 3), (3, 1), (3, 2)\}$ let $V_{\varepsilon,n}^{i_1,j_1}(\mathbf{a}) = v_{\varepsilon,n}^{-2} U_{\varepsilon,n}^{i_1,j_1}(\mathbf{a})$, $V_{\varepsilon,n}^{i_2,j_2}(\mathbf{a}) = v_{\varepsilon,n}^{-1} n^{-\frac{1}{2}} U_{\varepsilon,n}^{i_2,j_2}(\mathbf{a})$ and

$$\begin{aligned} V_{\varepsilon,n}^{i_1,j_1}(\mathbf{a}) &= (-1)^{i_1+j_1} a_3^{-2} \int_0^1 y_0(t)^{i_1+j_1-2-\frac{2}{q}} \langle H_1(Y_0(\mathbf{a}, t, \cdot), p) \rangle dt, \\ V_{\varepsilon,n}^{i_2,j_2}(\mathbf{a}) &= (-1)^{i_2+j_2} a_3^{-2} \int_0^1 y_0(t)^{i_2+j_2-4-\frac{1}{q}} \langle H_2(Y_0(\mathbf{a}, t, \cdot), p) \rangle dt. \end{aligned}$$

Put $V_{\varepsilon,n}^{3,3}(\mathbf{a}) = n^{-1} U_{\varepsilon,n}^{3,3}(\mathbf{a})$ and $V^{3,3}(\mathbf{a}) := a_3^{-2} \langle H_3(Y_0(\mathbf{a}, t, \cdot), p) \rangle$. Then $\mathbf{V}(\mathbf{a}) = (V^{i,j}(\mathbf{a}))$. Define the matrix $\mathbf{V}_{\varepsilon,n}(\mathbf{a}) = (V_{\varepsilon,n}^{i,j}(\mathbf{a}))$. Set $\mathbf{\Lambda}_{\varepsilon,n} = (v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^1(\bar{\mathbf{a}}), v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^2(\bar{\mathbf{a}}), n^{-\frac{1}{2}} U_{\varepsilon,n}^3(\bar{\mathbf{a}}))$.

Lemma 3.3 Suppose that Conditions 2.1–2.2 hold. Then as $n \rightarrow \infty$,

$$\mathbf{\Lambda}_{\varepsilon,n} \xrightarrow{d} N(\mathbf{0}, \bar{a}_3^{-2} \mathbf{\Sigma}) \quad \text{and} \quad \sup_{\mathbf{a} \in \bar{\mathbf{A}}} |\mathbf{V}_{\varepsilon,n}(\mathbf{a}) - \mathbf{V}(\mathbf{a})| \xrightarrow{p} 0.$$

Proof. It is easy to see that

$$\begin{aligned} v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^1(\bar{\mathbf{a}}) &= -\bar{a}_3^{-1} n^{-\frac{1}{2}} \sum_{k=1}^n H_0(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) N_{\varepsilon,n,k}, \\ v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^2(\bar{\mathbf{a}}) &= \bar{a}_3^{-1} n^{-\frac{1}{2}} \sum_{k=1}^n H_0(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) M_{\varepsilon,n,k}, \\ n^{-\frac{1}{2}} U_{\varepsilon,n}^3(\bar{\mathbf{a}}) &= -\bar{a}_3^{-1} n^{-\frac{1}{2}} \sum_{k=1}^n [H_0(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) Y_{\varepsilon,n,k}(\bar{\mathbf{a}}) + 1]. \end{aligned}$$

Observe that $\langle H_0, p \rangle = 0$ and $\int_{\mathbb{R}} [xH_0(x) + 1]p(x)dx = 0$. Then by Lemmas 3.1 and 3.2,

$$x_1 v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^1(\bar{\mathbf{a}}) + x_2 v_{\varepsilon,n}^{-1} U_{\varepsilon,n}^2(\bar{\mathbf{a}}) + x_3 n^{-\frac{1}{2}} U_{\varepsilon,n}^3(\bar{\mathbf{a}}) \xrightarrow{d} N(0, \eta(x_1, x_2, x_3)^2)$$

for all $x_i \in \mathbb{R}$ ($i = 1, 2, 3$) as $n \rightarrow \infty$, where

$$\eta(x_1, x_2, x_3)^2 := \bar{a}_3^{-2} \int_0^1 dt \int_{\mathbb{R}} [(x_1 y_0(t)^{-\frac{1}{q}} - x_2 y_0(t)^{1-\frac{1}{q}}) p'(x) + x_3 (x p'(x) + p(x))]^2 / p(x) dx.$$

Then the first assertion follows from the Cramér-Wold theorem.

It is elementary to see that for $1 \leq i_1, j_1 \leq 2$ and $(i_2, j_2) \in \{(1, 3), (2, 3), (3, 1), (3, 2)\}$,

$$V_{\varepsilon,n}^{i_1,j_1}(\mathbf{a}) = (-1)^{i_1+j_1} a_3^{-2} n^{-1} \sum_{k=1}^n H_1(Y_{\varepsilon,n,k}(\mathbf{a})) M_{\varepsilon,n,k}^{i_1+j_1-2} N_{\varepsilon,n,k}^{4-i_1-j_1},$$

$$V_{\varepsilon,n}^{i_2,j_2}(\mathbf{a}) = (-1)^{i_2+j_2} a_3^{-2} n^{-1} \sum_{k=1}^n H_2(Y_{\varepsilon,n,k}(\mathbf{a})) M_{\varepsilon,n,k}^{i_2+j_2-4} N_{\varepsilon,n,k}^{5-i_2-j_2}$$

and $V_{\varepsilon,n}^{3,3}(\mathbf{a}) = \frac{a_3^{-2}}{n} \sum_{k=1}^n H_3(Y_{\varepsilon,n,k}(\mathbf{a}))$. It follows from Lemma 3.2 that $H_i \in C^1(\mathbb{R})$ satisfies (3.3) with H replaced by H_i for $i = 1, 2, 3$. Therefore, by Lemma 3.1 we know that $\sup_{\mathbf{a} \in \bar{\mathbf{A}}} |V_{\varepsilon,n}^{i,j}(\mathbf{a}) - V^{i,j}(\mathbf{a})| \xrightarrow{p} 0$ for each $1 \leq i, j \leq 3$, which derives the last assertion. \square

Proof of Theorem 2.3. The proof is a modification of that of [19, Theorem 1]. We give some details in the following. Suppose that there exists a subsequence (ε_{n_k}, n_k) so that $\hat{\mathbf{a}}_{\varepsilon_{n_k}, n_k}$ tends to a limit $\check{\mathbf{a}} = (\check{a}_1, \check{a}_2, \check{a}_3)$. Taking $H(x) = \log p(x)$ in Lemma 3.1, we get

$$\sup_{\mathbf{a} \in \bar{\mathbf{A}}} |n^{-1} U_{\varepsilon,n}(\mathbf{a}) - U(\mathbf{a})| \xrightarrow{p} 0, \quad (3.6)$$

where $U(\mathbf{a})$ is defined in (2.1). By (1.4), for each $k \geq 1$ we get $\frac{1}{n_k} U_{\varepsilon_{n_k}, n_k}(\bar{\mathbf{a}}) \leq \frac{1}{n_k} U_{\varepsilon_{n_k}, n_k}(\hat{\mathbf{a}}_{\varepsilon_{n_k}, n_k})$. Letting $k \rightarrow \infty$, by (3.6), we have $U(\bar{\mathbf{a}}) \leq U(\check{\mathbf{a}})$. On the other hand, $\bar{\mathbf{a}}$ is the only maximum point of $U(\mathbf{a})$ by Condition 2.2(i), thus $\check{\mathbf{a}} = \bar{\mathbf{a}}$. This proves (2.2).

By Taylor's formula, $\mathbf{S}_{\varepsilon,n} \mathbf{D}_{\varepsilon,n} = \mathbf{\Lambda}_{\varepsilon,n}$, where $\mathbf{D}_{\varepsilon,n} = \int_0^1 \mathbf{V}_{\varepsilon,n}(\bar{\mathbf{a}} + u(\hat{\mathbf{a}}_{\varepsilon,n} - \bar{\mathbf{a}})) du$. Then by Lemma 3.3 and the fact $\mathbf{V}(\bar{\mathbf{a}}) = -\bar{a}_3^{-2} \mathbf{\Sigma}$, one obtains (2.3) by using the same argument in the corresponding proof of [19, Theorem 1]. \square

4 Appendix

Lemma 4.1 *Suppose that Condition 2.1 holds. Then $\mathbf{\Sigma}$ is a positive definite matrix.*

Proof. By the Hölder inequality, the determinant

$$\begin{vmatrix} v_1 m^{0,2} & -v_1 m^{1,2} \\ -v_1 m^{1,2} & v_1 m^{2,2} \end{vmatrix} = v_1^2 [m^{0,2} m^{2,2} - (m^{1,2})^2] > 0 \quad (4.1)$$

and $1 = |\int_{\mathbb{R}} \frac{xp'(x)}{p(x)^{1/2}} p(x)^{1/2} dx|^2 < \int_{\mathbb{R}} [xp'(x)]^2 / p(x) dx$, which implies $v_3 > 0$. It is obvious that

$$|\mathbf{\Sigma}| v_3 = [v_1 v_3 m^{2,2} - (v_2 m^{1,1})^2] [v_1 v_3 m^{0,2} - (v_2 m^{0,1})^2] - [v_1 v_3 m^{1,2} - v_2^2 m^{0,1} m^{1,1}]^2. \quad (4.2)$$

Since $\langle 1, p' \rangle = 0$ and $\int_{\mathbb{R}} xp'(x) dx = -1$, by the Hölder inequality again we get

$$\begin{aligned} v_2^2 &= \left| \int_{\mathbb{R}} \frac{p'(x)}{p(x)} [xp'(x) + p(x)] dx \right|^2 < \int_{\mathbb{R}} \frac{|p'(x)|^2}{p(x)} dx \int_{\mathbb{R}} \frac{[xp'(x) + p(x)]^2}{p(x)} dx \\ &= \int_{\mathbb{R}} \frac{|p'(x)|^2}{p(x)} dx \left[\int_{\mathbb{R}} \frac{[xp'(x)]^2}{p(x)} dx + 2 \int_{\mathbb{R}} xp'(x) dx + 1 \right] = v_1 v_3. \end{aligned}$$

Note that $|\int_0^1 [y_0(t)^{1-1/q} z + y_0(t)^{-1/q}] dt|^2 < \int_0^1 [y_0(t)^{1-1/q} z + y_0(t)^{-1/q}]^2 dt$ for each $z \in \mathbb{R}$. Then

$$\begin{aligned} &[v_1 v_3 m^{2,2} - (v_2 m^{1,1})^2] z^2 + 2[v_1 v_3 m^{1,2} - v_2^2 m^{0,1} m^{1,1}] z + [v_1 v_3 m^{0,2} - (v_2 m^{0,1})^2] \\ &= v_1 v_3 [m^{2,2} z^2 + 2zm^{1,2} + m^{0,2}] - v_2^2 [(m^{1,1} z)^2 + 2zm^{1,1} m^{0,1} + (m^{0,1})^2] \\ &= v_1 v_3 \int_0^1 [y_0(t)^{1-1/q} z + y_0(t)^{-1/q}]^2 dt - v_2^2 \left| \int_0^1 [y_0(t)^{1-1/q} z + y_0(t)^{-1/q}] dt \right|^2 > 0 \end{aligned}$$

for all $z \in \mathbb{R}$. It follows from (4.2) that $|\Sigma|v_3 > 0$, which implies $|\Sigma| > 0$. Together with (4.1) one gets the desired result. \square

Proof of Lemma 3.1. In the following C is a constant whose value might change from place to place and does not depend on ε, n, k, t and \mathbf{a} . For $\mathbf{a} \in \bar{\mathbf{A}}$, $n, k \geq 1$, $\varepsilon > 0$ and $t \in [0, 1]$ we put $B_{\varepsilon, n, k} := B(M_{\varepsilon, n, k}, N_{\varepsilon, n, k})$, $Y_{n, k}(\mathbf{a}, t) := m_0 Y(\mathbf{a}, t) + \bar{a}_3 a_3^{-1} K_{n, k}$ and $\bar{Y}_{n, k}(\mathbf{a}, t, \varepsilon) := m_{\varepsilon, n} Y(\mathbf{a}, t) + \bar{a}_3 a_3^{-1} K_{n, k}$. For $0 < \zeta < \inf_{t \in [0, 1]} y_0(t)$ and $\varsigma > \|y_0\|$ define $A_{\varepsilon, \zeta} = \{\inf_{t \in [0, 1]} y_\varepsilon(t) \geq \zeta\}$ and $B_{\varepsilon, \varsigma} = \{\sup_{t \in [0, 1]} y_\varepsilon(t) \leq \varsigma\}$. It follows from [17, Lemma 3.5] that

$$\mathbf{P}\{A_{\varepsilon, \zeta}^c\} + \mathbf{P}\{B_{\varepsilon, \varsigma}^c\} \leq C\varepsilon^\alpha. \quad (4.3)$$

Define $U_{\varepsilon, \varsigma, \zeta} = A_{\varepsilon, \zeta} \cap B_{\varepsilon, \varsigma}$. We divide the rest of proof into seven steps.

Step 1. First we show: For each $\gamma' > 1$ and large enough $n_0 > 0$,

$$\sup_{n, k \geq 1, \varepsilon > 0} \mathbf{E} \left\{ \sup_{\mathbf{a} \in \bar{\mathbf{A}}} [-Y_{\varepsilon, n, k}(\mathbf{a})]^{\gamma'} 1_{\{Y_{\varepsilon, n, k}(\mathbf{a}) < -n_0, U_{\varepsilon, \varsigma, \zeta}\}} \right\} < \infty, \quad (4.4)$$

$$\sup_{n, k \geq 1} \mathbf{E} \left\{ [-K_{n, k}]^{\gamma'} 1_{\{K_{n, k} < 0\}} \right\} < \infty. \quad (4.5)$$

By [8, Theorem 1.4], for each $\varepsilon > 0$ and $n, k \geq 1$ there is a stable process $\{z_{\varepsilon, n, k}(t) : t \geq 0\}$ with the same finite dimension distribution as $\{z_0(t) : t \geq 0\}$ so that $K_{\varepsilon, n, k} = z_{\varepsilon, n, k}(T_{\varepsilon, n, k})$, where $T_{\varepsilon, n, k} = n \int_{t_{k-1}}^{t_k} [y_\varepsilon(s) y_\varepsilon([ns]/n)^{-1}]^{\alpha/q} ds$. Observe that $T_{\varepsilon, n, k} \leq (\varsigma \zeta^{-1})^{\alpha/q}$ on $A_{\varepsilon, \zeta} \cap B_{\varepsilon, \varsigma}$. It follows from [3, Lemma 2.4] that for each $x > 0$,

$$\mathbf{P}\{K_{\varepsilon, n, k} \leq -x, U_{\varepsilon, \varsigma, \zeta}\} \leq \mathbf{P}\left\{ \inf_{t \leq \varsigma \zeta^{-1}} z_{\varepsilon, n, k}(t) \leq -x \right\} \leq \exp\{-\tilde{c}_0 x^{\alpha/(\alpha-1)}\}, \quad (4.6)$$

where $\tilde{c}_0 := [(\alpha-1)/\alpha]^{\alpha/(\alpha-1)} [\zeta \zeta^{-1}]^{\frac{\alpha}{q(\alpha-1)}}$. One can also see that $\bar{M}_{\varepsilon, n, k}(a_2) \leq |a_2| \varsigma^{1-1/q} + |\bar{a}_2| \varsigma \zeta^{-1/q}$ and $N_{\varepsilon, n, k} \leq \zeta^{-1/q}$ on $U_{\varepsilon, \varsigma, \zeta}$, which implies $|Y_{\varepsilon, n, k}(\mathbf{a})|^{\gamma'} \leq \tilde{c}_1 + |2\bar{a}_3 a_3^{-1} K_{\varepsilon, n, k}|^{\gamma'}$ on $U_{\varepsilon, \varsigma, \zeta}$ with

$$\tilde{c}_1 := \sup_{\mathbf{a} \in \bar{\mathbf{A}}, \varepsilon > 0, n \geq 1} |2m_{\varepsilon, n} a_3^{-1} [|a_2| \varsigma^{1-1/q} + |\bar{a}_2| \varsigma \zeta^{-1/q} + |a_1 - \bar{a}_1| \zeta^{-1/q}]|^{\gamma'}.$$

Thus

$$\begin{aligned} \mathbf{E} \left\{ \sup_{\mathbf{a} \in \bar{\mathbf{A}}} [-Y_{\varepsilon, n, k}(\mathbf{a})]^{\gamma'} 1_{\{Y_{\varepsilon, n, k}(\mathbf{a}) < -n_0, U_{\varepsilon, \varsigma, \zeta}\}} \right\} &\leq C \mathbf{E} \left\{ [-K_{\varepsilon, n, k}]^{\gamma'} 1_{\{K_{\varepsilon, n, k} < 0, U_{\varepsilon, \varsigma, \zeta}\}} \right\} + \tilde{c}_1 \\ &= C \int_0^\infty t^{\gamma'-1} \mathbf{P}\{K_{\varepsilon, n, k} < -t, U_{\varepsilon, \varsigma, \zeta}\} dt + \tilde{c}_1 \end{aligned} \quad (4.7)$$

for large enough n_0 . Together with (4.6) implies (4.4). Similarly, one can also get (4.5).

Step 2. In this step we show that for $\delta \in (1, \alpha)$ and $\delta' := \delta/(\delta-1)$,

$$\begin{aligned} \mathbf{E} \left\{ \sup_{\mathbf{a} \in \bar{\mathbf{A}}} |H(Y_{\varepsilon, n, k}(\mathbf{a})) B_{\varepsilon, n, k} - H(\bar{Y}_{n, k}(\mathbf{a}, t, \varepsilon)) B_t| 1_{U_{\varepsilon, \varsigma, \zeta}} \right\} \\ \leq C \left\{ \mathbf{E} [|K_{\varepsilon, n, k} - K_{n, k}|^\delta 1_{U_{\varepsilon, \varsigma, \zeta}}] \right\}^{1/\delta} \\ + C \left\{ \mathbf{E} [| \tilde{M}_{\varepsilon, n, k, 1}(t) | + | \tilde{M}_{\varepsilon, n, k, 2}(t) | + | \tilde{N}_{\varepsilon, n, k}(t) |]^\delta 1_{U_{\varepsilon, \varsigma, \zeta}} \right\}^{1/\delta'} \end{aligned} \quad (4.8)$$

for $t \in (t_{k-1}, t_k]$ and

$$\mathbf{E} \left\{ |H(Y_{\varepsilon, n, k}(\bar{\mathbf{a}})) B_{\varepsilon, n, k} - H(K_{n, k}) B_{\varepsilon, n, k}| 1_{U_{\varepsilon, \varsigma, \zeta}} \right\}$$

$$\leq C \left\{ \mathbf{E} \left[\left| \bar{M}_{\varepsilon,n,k}(\bar{a}_2) \right|^\delta + |K_{\varepsilon,n,k} - K_{n,k}|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right] \right\}^{1/\delta}, \quad (4.9)$$

where

$$\tilde{N}_{\varepsilon,n,k}(t) := N_{\varepsilon,n,k} - y_0(t)^{-\frac{1}{q}}, \quad \tilde{M}_{\varepsilon,n,k,1}(t) = n \int_{t_{k-1}}^{t_k} y_\varepsilon([ns]/n)^{1-1/q} ds - y_0(t)^{1-1/q}, \quad (4.10)$$

$$\tilde{M}_{\varepsilon,n,k,2}(t) := n \int_{t_{k-1}}^{t_k} y_\varepsilon(s) y_\varepsilon([ns]/n)^{-1/q} ds - y_0(t)^{1-1/q}. \quad (4.11)$$

For $n \geq 2$ define functions $H_n(x) = H(x)1_{\{x > -n\}}$ and $G_n(x) = H(x)1_{\{x \leq -n\}}$. Then $|H(x) - H(y)| \leq |H_n(x) - H_n(y)| + |G_n(x) - G_n(y)|$ for $n \geq 4$ and $x, y \in \mathbb{R}$. Let $\tilde{H}_n, \tilde{G}_n \in C^1(\mathbb{R})$ satisfy $\tilde{H}_n(x) = H_n(x)$ for all $x \in (-\infty, -n-1) \cup (-n, \infty)$ and $\tilde{G}_n(x) = G_n(x)$ for all $x \in (-\infty, -n) \cup (-n+1, \infty)$. For large enough n_1 , $n_2 := n_1 + 2$ and $n_3 := n_1 + 4$, we have $|H(x) - H(y)| \leq \sum_{k=1}^3 [|\tilde{H}_{n_k}(x) - \tilde{H}_{n_k}(y)| + |\tilde{G}_{n_k}(x) - \tilde{G}_{n_k}(y)|]$. Then by (3.3) and the mean value theorem, there is a constant $\tilde{c}_2 = \tilde{c}_2(n_1) > 0$ so that $|\tilde{H}_{n_k}(x) - \tilde{H}_{n_k}(y)| \leq \tilde{c}_2|x - y|$ and

$$|\tilde{G}_{n_k}(x) - \tilde{G}_{n_k}(y)| \leq |x - y| \int_0^1 |\tilde{G}'_{n_k}(x + h(y - x))| dh \leq \tilde{c}_2 [|x|^\gamma 1_{\{x < 0\}} + |y|^\gamma 1_{\{y < 0\}}] |x - y|$$

for $x, y \in \mathbb{R}$. It thus follows that

$$|H(x) - H(y)| \leq \tilde{c}_2 [1 + |x|^\gamma 1_{\{x < 0\}} + |y|^\gamma 1_{\{y < 0\}}] |x - y|, \quad x, y \in \mathbb{R}. \quad (4.12)$$

Since $1/\delta + 1/\delta' = 1$, by the Hölder inequality we have

$$\begin{aligned} & \mathbf{E} \left\{ \sup_{\mathbf{a} \in \bar{\mathbf{A}}} |H(Y_{\varepsilon,n,k}(\mathbf{a})) - H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))| 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \\ & \leq \left\{ \mathbf{E} \left[\sup_{\mathbf{a} \in \bar{\mathbf{A}}} [1 + |Y_{\varepsilon,n,k}(\mathbf{a})|^\gamma 1_{\{Y_{\varepsilon,n,k}(\mathbf{a}) < 0\}} + |\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)|^\gamma 1_{\{\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon) < 0\}}]^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right] \right\}^{\frac{1}{\delta'}} \\ & \quad \cdot \left\{ \mathbf{E} \left[\sup_{\mathbf{a} \in \bar{\mathbf{A}}} |Y_{\varepsilon,n,k}(\mathbf{a}) - \bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right] \right\}^{1/\delta}, \quad t_{k-1} < t \leq t_k. \end{aligned} \quad (4.13)$$

Observe that

$$\sup_{n,k \geq 1, \varepsilon > 0, t \in [0,1]} [|\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)| + |\tilde{N}_{\varepsilon,n,k}(t)| + |M_{\varepsilon,n,k}| + |N_{\varepsilon,n,k}|] 1_{U_{\varepsilon,\varsigma,\zeta}} < \infty.$$

Then by the fact $\delta' > \delta$ and Hölder inequality again we get

$$\begin{aligned} & \mathbf{E} \left\{ |\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)| [|\tilde{N}_{\varepsilon,n,k}(t)| + |\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)|] 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \\ & \leq \left\{ \mathbf{E} [|\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)|^\delta] \right\}^{\frac{1}{\delta}} \left\{ \mathbf{E} [|\tilde{N}_{\varepsilon,n,k}(t)| + |\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)|]^{\delta'} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\}^{\frac{1}{\delta'}} \\ & \leq C \left\{ \mathbf{E} [|\tilde{N}_{\varepsilon,n,k}(t)| + |\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)|]^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right\}^{\frac{1}{\delta}}. \end{aligned} \quad (4.14)$$

Since $B \in C(\mathbb{R}_+^2)$, $\sup_{n,k \geq 1, \varepsilon > 0} B_{\varepsilon,n,k} 1_{U_{\varepsilon,\varsigma,\zeta}} < \infty$. In view of (4.12),

$$|H(x)| \leq |H(x) - H(0)| + |H(0)| \leq C(|x|^{\gamma+1} 1_{\{x < 0\}} + x 1_{\{x > 0\}}) + C, \quad (4.15)$$

which derives

$$|H(Y_{\varepsilon,n,k}(\mathbf{a})) B_{\varepsilon,n,k} - H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)) B_t|$$

$$\begin{aligned}
&= |H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))[B_{\varepsilon,n,k} - B_t] + B_{\varepsilon,n,k}[H(Y_{\varepsilon,n,k}(\mathbf{a})) - H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))]| \\
&\leq C \left[1 + |\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)|^{\gamma+1} 1_{\{\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon) < 0\}} + \bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon) 1_{\{\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon) > 0\}} \right] \cdot \left[|\tilde{N}_{\varepsilon,n,k}(t)| \right. \\
&\quad \left. + [|a_2| + |\bar{a}_2|][|\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)|] \right] + C |H(Y_{\varepsilon,n,k}(\mathbf{a})) - H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))|
\end{aligned}$$

on $U_{\varepsilon,\varsigma,\zeta}$ and

$$\begin{aligned}
|Y_{\varepsilon,n,k}(\mathbf{a}) - \bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)| &\leq |a_3^{-1} [|a_2| + |\bar{a}_2|] m_{\varepsilon,n} [|\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)|] \\
&\quad + |(\bar{a}_1 - a_1) a_3^{-1} m_{\varepsilon,n} |\tilde{N}_{\varepsilon,n,k}(t)| + |\bar{a}_3 a_3^{-1} |K_{\varepsilon,n,k} - K_{n,k}|.
\end{aligned}$$

Together with (4.4)–(4.5), (4.13)–(4.14) and Condition 2.2(ii) one can get (4.8). By using (4.12),

$$\begin{aligned}
&|H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) - H(K_{n,k})| \\
&\leq C \left[1 + |Y_{\varepsilon,n,k}(\bar{\mathbf{a}})|^{\gamma} 1_{\{Y_{\varepsilon,n,k}(\bar{\mathbf{a}}) < 0\}} + |K_{n,k}|^{\gamma} 1_{\{K_{n,k} < 0\}} \right] [|\bar{M}_{\varepsilon,n,k}(\bar{a}_2)| + |K_{\varepsilon,n,k} - K_{n,k}|].
\end{aligned}$$

Thus (4.9) follows from (4.4)–(4.5) and the Hölder inequality.

Step 3. Let $\delta'' \geq 1$, $\delta \in (1, \alpha)$ and $0 \leq t' \leq t'' \leq 1$ with $|t' - t''| \leq 1/n$. Now we show

$$\mathbf{E} \left\{ |y_{\varepsilon}(t')^{1/\delta''} - y_0(t'')^{1/\delta''}|^{\delta} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \leq C [\varepsilon n^{-1/\alpha} + n^{-1} + \varepsilon]^{\delta}. \quad (4.16)$$

By using Itô's formula on (1.2) one derives that

$$e^{\bar{a}_2 t} y_{\varepsilon}(t) = x_0 + \bar{a}_1 \int_0^t e^{\bar{a}_2 s} ds + \varepsilon \bar{a}_3 \int_0^t e^{\bar{a}_2 s} y_{\varepsilon}(s-)^{1/q} dz_0(s), \quad t \in [0, 1]. \quad (4.17)$$

For $s \in [0, 1]$ define $E_s = \{y_{\varepsilon}([ns]/n) \leq \varsigma, y_{\varepsilon}(s-) \leq \varsigma\}$. Then on $U_{\varepsilon,\varsigma,\zeta}$,

$$\begin{aligned}
|y_{\varepsilon}(t') - y_{\varepsilon}(t'')| &\leq e^{-\bar{a}_2 t''} |e^{\bar{a}_2 t'} y_{\varepsilon}(t') - e^{\bar{a}_2 t''} y_{\varepsilon}(t'')| + |1 - e^{\bar{a}_2(t' - t'')}| \varsigma \\
&\leq |\varepsilon \bar{a}_3| \left| \int_{t'}^{t''} e^{\bar{a}_2(s - t'')} y_{\varepsilon}(s-)^{1/q} 1_{E_s} dz_0(s) \right| + e^{|\bar{a}_2|} (|\bar{a}_1| + \varsigma) n^{-1}.
\end{aligned}$$

It follows from [14, Lemma 4.4] that for each $0 < \delta_0 < \alpha$,

$$\mathbf{E} \left\{ |y_{\varepsilon}(t') - y_{\varepsilon}(t'')|^{\delta_0} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \leq C \mathbf{E} \left\{ \left[\int_{t'}^{t''} \varepsilon y_{\varepsilon}(s)^{\frac{\alpha}{q}} 1_{E_s} ds \right]^{\frac{\delta_0}{\alpha}} \right\} + \frac{C}{n^{\delta_0}} \leq C [\varepsilon n^{-1/\alpha} + n^{-1}]^{\delta_0}. \quad (4.18)$$

By (4.17) and [14, Lemma 4.4] again, for each $t \in [0, 1]$,

$$\mathbf{E} \left\{ |y_{\varepsilon}(t) - y_0(t)|^{\delta} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \leq \mathbf{E} \left[\left| \varepsilon \bar{a}_3 \int_0^t e^{\bar{a}_2(s-t)} y_{\varepsilon}(s-)^{1/q} 1_{E_s} dz_0(s) \right|^{\delta} \right] \leq C \varepsilon^{\delta}. \quad (4.19)$$

Observe that $|z_1^{1/\delta''} - z_2^{1/\delta''}| \leq C |z_1 - z_2|$ for $z_1, z_2 \in [\zeta, \varsigma]$. Together with (4.18)–(4.19) we have

$$\begin{aligned}
&\mathbf{E} \left\{ |y_{\varepsilon}(t')^{1/\delta''} - y_0(t'')^{1/\delta''}|^{\delta} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \\
&\leq 2^{\delta} \mathbf{E} \left\{ |y_{\varepsilon}(t')^{1/\delta''} - y_{\varepsilon}(t'')^{1/\delta''}|^{\delta} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} + 2^{\delta} \mathbf{E} \left\{ |y_{\varepsilon}(t'')^{1/\delta''} - y_0(t'')^{1/\delta''}|^{\delta} 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \\
&\leq C \mathbf{E} \left\{ \left[|y_{\varepsilon}(t') - y_{\varepsilon}(t'')|^{\delta} + |y_{\varepsilon}(t'') - y_0(t'')|^{\delta} \right] 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \leq C [\varepsilon n^{-1/\alpha} + n^{-1} + \varepsilon]^{\delta},
\end{aligned}$$

which derives (4.16).

Step 4. Recall (3.2) and (4.10)–(4.11). Let $1 < \delta < \alpha$. In this step we show for $t \in (t_{k-1}, t_k]$,

$$\mathbf{E} \left[\left| |\tilde{M}_{\varepsilon,n,k,1}(t)| + |\tilde{M}_{\varepsilon,n,k,2}(t)| + |\tilde{N}_{\varepsilon,n,k}(t)| \right|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right] \leq C[\varepsilon n^{-1/\alpha} + n^{-1} + \varepsilon]^\delta, \quad (4.20)$$

$$\mathbf{E} \{ |\bar{M}_{\varepsilon,n,k}(\bar{a}_2)|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \} \leq C[\varepsilon n^{-1/\alpha} + n^{-1}]^\delta, \quad (4.21)$$

$$\mathbf{E} \{ |K_{\varepsilon,n,k} - K_{n,k}|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \} \leq C[\varepsilon n^{-1/\alpha} + n^{-1}]^{r\delta}, \quad r \in (0, 1). \quad (4.22)$$

Observe that on $U_{\varepsilon,\varsigma,\zeta}$, we have $|\tilde{M}_{\varepsilon,n,k,1}(t)|^\delta \leq n \int_{t_{k-1}}^{t_k} |y_\varepsilon([ns]/n)^{1-1/q} - y_0(t)^{1-1/q}|^\delta ds$,

$$\begin{aligned} |\tilde{M}_{\varepsilon,n,k,2}(t)|^\delta &\leq \left| n \int_{t_{k-1}}^{t_k} [y_\varepsilon(s) - y_\varepsilon([ns]/n)] y_\varepsilon([ns]/n)^{-1/q} ds + \tilde{M}_{\varepsilon,n,k,1}(t) \right|^\delta \\ &\leq 2^\delta y_\varepsilon(t_{k-1})^{-\delta/q} n \int_{t_{k-1}}^{t_k} |y_\varepsilon(s) - y_\varepsilon([ns]/n)|^\delta ds + 2^\delta |\tilde{M}_{\varepsilon,n,k,1}(t)|^\delta \\ &\leq 2^\delta \zeta^{-\delta/q} n \int_{t_{k-1}}^{t_k} |y_\varepsilon(s) - y_\varepsilon([ns]/n)|^\delta ds + 2^\delta |\tilde{M}_{\varepsilon,n,k,1}(t)|^\delta, \\ |\tilde{N}_{\varepsilon,n,k}(t)|^\delta &\leq n \int_{t_{k-1}}^{t_k} |y_\varepsilon([ns]/n)^{-1/q} - y_0(t)^{-1/q}|^\delta ds \\ &\leq y_0(t)^{-\delta/q} \zeta^{-\delta/q} n \int_{t_{k-1}}^{t_k} |y_\varepsilon([ns]/n)^{1/q} - y_0(t)^{1/q}|^\delta ds \end{aligned}$$

and $|\bar{M}_{\varepsilon,n,k}(\bar{a}_2)|^\delta \leq |\bar{a}_2|^\delta \zeta^{-\delta/q} n \int_{t_{k-1}}^{t_k} |y_\varepsilon([ns]/n) - y_\varepsilon(s)|^\delta ds$. Then (4.20) and (4.21) follows from (4.16) and (4.18). Observe that for $t', t'' \in [0, 1]$,

$$[y_\varepsilon(t') y_\varepsilon(t'')^{-1}]^{1/q} - 1 = y_\varepsilon(t'')^{-1-\frac{1}{q}} \left\{ [y_\varepsilon(t'') - y_\varepsilon(t')] y_\varepsilon(t')^{\frac{1}{q}} - [y_\varepsilon(t'')^{1+\frac{1}{q}} - y_\varepsilon(t')^{1+\frac{1}{q}}] \right\}.$$

Recall that $E_s = \{y_\varepsilon([ns]/n) \leq \varsigma, y_\varepsilon(s-) \leq \varsigma\}$ for $s \in [0, 1]$. Then on $U_{\varepsilon,\varsigma,\zeta}$,

$$\begin{aligned} |K_{\varepsilon,n,k} - K_{n,k}|^\delta &= \left| n^{1/\alpha} \int_{t_{k-1}}^{t_k} \left[[y_\varepsilon(s-) y_\varepsilon([ns]/n)^{-1}]^{1/q} - 1 \right] dz_0(s) \right|^\delta \\ &\leq 2^\delta y_\varepsilon(t_{k-1})^{-\delta-\frac{\delta}{q}} \left| n^{1/\alpha} \int_{t_{k-1}}^{t_k} \left[[y_\varepsilon([ns]/n) - y_\varepsilon(s-)] y_\varepsilon(s-)^{\frac{1}{q}} \right] dz_0(s) \right|^\delta \\ &\quad + 2^\delta y_\varepsilon(t_{k-1})^{-\delta-\frac{\delta}{q}} \left| n^{1/\alpha} \int_{t_{k-1}}^{t_k} \left[y_\varepsilon([ns]/n)^{1+\frac{1}{q}} - y_\varepsilon(s-)^{1+\frac{1}{q}} \right] dz_0(s) \right|^\delta \\ &\leq 2^\delta \zeta^{-\delta-\frac{\delta}{q}} \left| n^{1/\alpha} \int_{t_{k-1}}^{t_k} \left[y_\varepsilon([ns]/n) - y_\varepsilon(s-) \right] y_\varepsilon(s-)^{\frac{1}{q}} 1_{E_s} dz_0(s) \right|^\delta \\ &\quad + 2^\delta \zeta^{-\delta-\frac{\delta}{q}} \left| n^{1/\alpha} \int_{t_{k-1}}^{t_k} \left[y_\varepsilon([ns]/n)^{1+\frac{1}{q}} - y_\varepsilon(s-)^{1+\frac{1}{q}} \right] 1_{E_s} dz_0(s) \right|^\delta. \end{aligned}$$

Thus by [14, Lemma 4.4] and Jensen's inequality again,

$$\begin{aligned} &\mathbf{E} \left\{ |K_{\varepsilon,n,k} - K_{n,k}|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}} \right\} \\ &\leq C \mathbf{E} \left\{ \left[n \int_{t_{k-1}}^{t_k} \left[|y_\varepsilon([ns]/n) - y_\varepsilon(s)|^\alpha y_\varepsilon(s)^{\frac{\alpha}{q}} + |y_\varepsilon([ns]/n)^{1+\frac{1}{q}} - y_\varepsilon(s)^{1+\frac{1}{q}}|^\alpha \right] 1_{E_s} ds \right]^{\delta/\alpha} \right\}. \end{aligned}$$

Since

$$|y_\varepsilon([ns]/n)^{1+\frac{1}{q}} - y_\varepsilon(s)^{1+\frac{1}{q}}| \leq (1 + 1/q) \varsigma^{1/q} |y_\varepsilon([ns]/n) - y_\varepsilon(s)| \leq C |y_\varepsilon([ns]/n) - y_\varepsilon(s)|^r$$

on E_s by the mean value theorem, we know that

$$\mathbf{E}\{|K_{\varepsilon,n,k} - K_{n,k}|^\delta 1_{U_{\varepsilon,\varsigma,\zeta}}\} \leq C \{n \int_{t_{k-1}}^{t_k} \mathbf{E}[|y_\varepsilon(s) - y_\varepsilon([ns]/n)|^{r\alpha} 1_{E_s}] ds\}^{\delta/\alpha}$$

by the Hölder inequality. Combining with (4.18) we get (4.22).

Step 5. In this step we prove the following assertions:

$$n^{-1/2} \sum_{k=1}^n |[H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) - H(K_{n,k})]B_{\varepsilon,n,k}| \xrightarrow{p} 0, \quad (4.23)$$

$$n^{-1} \sum_{k=1}^n \sup_{\mathbf{a} \in \bar{\mathbf{A}}} |H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - \int_0^1 H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t dt| \xrightarrow{p} 0. \quad (4.24)$$

It follows from (4.9) and (4.21)–(4.22) that for any $r \in (\alpha/2, 1)$

$$\mathbf{E}\left\{|[H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) - H(K_{n,k})]B_{\varepsilon,n,k}| 1_{U_{\varepsilon,\varsigma,\zeta}}\right\} \leq C[\varepsilon n^{-1/\alpha} + n^{-1}]^r.$$

Observe that

$$\begin{aligned} & \left| H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - \int_0^1 H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t dt \right| \\ & \leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t| dt. \end{aligned}$$

Combining this with (4.8) and (4.20)–(4.22) we obtain that for each $\delta \in (1, \alpha)$ and $r \in (0, 1)$,

$$\mathbf{E}\left\{\sup_{\mathbf{a} \in \bar{\mathbf{A}}} \left| H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - \int_0^1 H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t dt \right| 1_{U_{\varepsilon,\varsigma,\zeta}}\right\} \leq C[\varepsilon n^{-1/\alpha} + n^{-1} + \varepsilon]^{r(\delta-1)}.$$

It thus follows from the Markov inequality that for each $\eta > 0$ and $r \in (\alpha/2, 1)$,

$$\begin{aligned} & \mathbf{P}\left\{n^{-1/2} \sum_{k=1}^n |[H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) - H(K_{n,k})]B_{\varepsilon,n,k}| > \eta, U_{\varepsilon,\varsigma,\zeta}\right\} \\ & \leq \eta^{-1} n^{-1/2} \sum_{k=1}^n \mathbf{E}\left\{|[H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}})) - H(K_{n,k})]B_{\varepsilon,n,k}| 1_{U_{\varepsilon,\varsigma,\zeta}}\right\} \\ & \leq C\eta^{-1} [n^{1/2-r} + \varepsilon^r n^{(\alpha-2r)/(2\alpha)}] \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \mathbf{P}\left\{\frac{1}{n} \sum_{k=1}^n \sup_{\mathbf{a} \in \bar{\mathbf{A}}} \left| H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - \int_0^1 H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t dt \right| > \eta, U_{\varepsilon,\varsigma,\zeta}\right\} \\ & \leq \eta^{-1} n^{-1} \sum_{k=1}^n \mathbf{E}\left\{\sup_{\mathbf{a} \in \bar{\mathbf{A}}} \left| H(Y_{\varepsilon,n,k}(\mathbf{a}))B_{\varepsilon,n,k} - \int_0^1 H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon))B_t dt \right| 1_{U_{\varepsilon,\varsigma,\zeta}}\right\} \\ & \leq C\eta^{-1} [\varepsilon n^{-1/\alpha} + n^{-1} + \varepsilon]^{r(\delta-1)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Then (4.23) and (4.24) follow from (4.3).

Step 6. In this step we prove (3.4). Since $\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon) - Y_{n,k}(\mathbf{a}, t) = (m_{\varepsilon,n} - m_0)Y(\mathbf{a}, t)$ and H is uniformly continuous on any bounded interval, we have as $n \rightarrow \infty$,

$$\sup_{\mathbf{a} \in \bar{\mathbf{A}}} \int_0^1 |[H(\bar{Y}_{n,k}(\mathbf{a}, t, \varepsilon)) - H(Y_{n,k}(\mathbf{a}, t))]B_t| dt \rightarrow 0. \quad (4.25)$$

By using (4.5) and (4.15), we get $\mathbf{E}\{\sup_{\mathbf{a} \in \bar{\mathbf{A}}} \int_0^1 |H(Y_{n,k}(\mathbf{a}, t))B_t| dt\} < \infty$. It is easy to see that for each $n, k \geq 1$ and $\mathbf{a} \in \bar{\mathbf{A}}$, we have $\mathbf{E}\{\int_0^1 H(Y_{n,k}(\mathbf{a}, t))B_t dt\} = \int_0^1 \langle H(Y_0(\mathbf{a}, t, \cdot)), p \rangle B_t dt$. Then by the uniform laws of large numbers (see e.g. [1, Theorem 4.2.1]),

$$\sup_{\mathbf{a} \in \bar{\mathbf{A}}} |n^{-1} \sum_{k=1}^n \int_0^1 H(Y_{n,k}(\mathbf{a}, t))B_t dt - \int_0^1 \langle H(Y_0(\mathbf{a}, t, \cdot)), p \rangle B_t dt| \xrightarrow{p} 0$$

as $n \rightarrow \infty$, Hence (3.4) follows from (4.24) and (4.25).

Step 7. In this step we show (3.5). Applying (3.4) we obtain

$$\frac{1}{n} \sum_{k=1}^n [H(K_{n,k})B_{\varepsilon,n,k} + \bar{H}(K_{n,k})]^2 \xrightarrow{p} \eta_0^2$$

as $n \rightarrow \infty$. Then by using [6, Theorem 3.4] one can see that

$$n^{-1/2} \sum_{k=1}^n [H(K_{n,k})B_{\varepsilon,n,k} + \bar{H}(K_{n,k})] \xrightarrow{d} N(0, \eta_0^2) \quad (4.26)$$

as $n \rightarrow \infty$. Applying (4.23) we obtain

$$n^{-1/2} \sum_{k=1}^n \left| [H(Y_{\varepsilon,n,k}(\bar{\mathbf{a}}))B_{\varepsilon,n,k} + \bar{H}(Y_{\varepsilon,n,k}(\bar{\mathbf{a}}))] - [H(K_{n,k})B_{\varepsilon,n,k} + \bar{H}(K_{n,k})] \right| \xrightarrow{p} 0$$

as $n \rightarrow \infty$. This, together with (4.26), implies (3.5), which completes the proof. \square

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